

# THE UNIQUENESS OF THE ENNEPER SURFACES AND CHERN-RICCI FUNCTIONS ON MINIMAL SURFACES

HOJOO LEE

**ABSTRACT.** We construct the first and second Chern-Ricci functions on negatively curved minimal surfaces in  $\mathbb{R}^3$  using Gauss curvature and angle functions, and establish that they become harmonic functions on the minimal surfaces. We prove that a minimal surface has constant first Chern-Ricci function if and only if it is Enneper's surface. We explicitly determine the moduli space of minimal surfaces having constant second Chern-Ricci function, which contains catenoids, helicoids, and their associate families.

*To my mother who always gives me courage*

## 1. ENNEPER'S SURFACES AND OTHER MINIMAL SURFACES IN $\mathbb{R}^3$

What is the Enneper surface? The Enneper-Weierstrass representation [9] says that a simply connected minimal surface in  $\mathbb{R}^3$  can be constructed by the conformal harmonic mapping

$$\mathbf{X}(\zeta) = \mathbf{X}(\zeta_0) + \left( \operatorname{Re} \int_{\zeta_0}^{\zeta} \phi_1(\zeta) d\zeta, \operatorname{Re} \int_{\zeta_0}^{\zeta} \phi_2(\zeta) d\zeta, \operatorname{Re} \int_{\zeta_0}^{\zeta} \phi_3(\zeta) d\zeta \right),$$

where the holomorphic null curve  $\phi(\zeta)$  is determined by the Weierstrass data  $(G(\zeta), \Psi(\zeta)d\zeta)$ :

$$\phi(\zeta) = (\phi_1(\zeta), \phi_2(\zeta), \phi_3(\zeta)) = \left( \frac{1}{2} (1 - G^2) \Psi, \frac{i}{2} (1 + G^2) \Psi, G\Psi \right)$$

Taking the simplest data  $(G(\zeta), \Psi(\zeta)d\zeta) = (\zeta, d\zeta)$ ,  $\zeta = u + iv \in \mathbb{C}$  yields Enneper's surface

$$(1.1) \quad \mathbf{X}_{\text{Enn}}(u, v) = \left( \frac{1}{2} \left( u - \frac{u^3}{3} + uv^2 \right), \frac{1}{2} \left( -v + \frac{v^3}{3} - u^2v \right), \frac{1}{2} (u^2 - v^2) \right).$$

One can see the *inner* rotational symmetry of its induced metric  $\mathbf{g}_{\text{Enn}} = \left( \frac{1+|\zeta|^2}{2} \right)^2 |d\zeta|^2$  and strictly negative Gauss curvature  $\mathcal{K}_{\text{Enn}} = \mathcal{K}_{\mathbf{g}_{\text{Enn}}} = -\left( \frac{2}{1+|\zeta|^2} \right)^4$ . We observe that

- (1) the conformally changed metric  $(-\mathcal{K}_{\text{Enn}})^{\frac{1}{2}} \mathbf{g}_{\text{Enn}} = |d\zeta|^2$  is flat, and that
- (2)  $(-\mathcal{K}_{\text{Enn}}) \mathbf{g}_{\text{Enn}} = \left( \frac{2}{1+|\zeta|^2} \right)^2 |d\zeta|^2$  becomes the metric on the complex  $\zeta$ -plane induced by the stereographic projection of the unit sphere sitting in  $\mathbb{R}^3$  with respect to the north pole.

Do these *intrinsic* properties uniquely determine Enneper's surfaces among minimal surfaces? The answer is completely no. Ricci [1, 3, 6, 7, 8] showed that, if  $\Sigma$  is a minimal surface in  $\mathbb{R}^3$ , on non-flat points, the metric  $(-\mathcal{K}_{\mathbf{g}_{\Sigma}})^{\frac{1}{2}} \mathbf{g}_{\Sigma}$  is flat. The Ricci condition guarantees that the metric  $(-\mathcal{K}_{\mathbf{g}_{\Sigma}}) \mathbf{g}_{\Sigma}$  has constant Gauss curvature 1. We see that Enneper's surface becomes the simplest example illustrating Ricci conditions for negatively curved minimal surfaces in  $\mathbb{R}^3$ .

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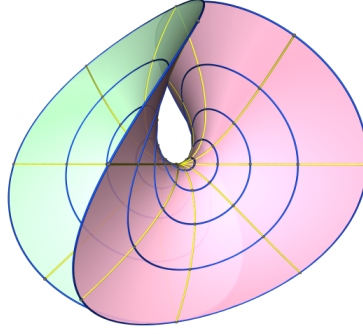


FIGURE 1. An approximation of a part of Enneper's surface with total curvature  $-4\pi$  [10]

We shall present the *coordinate-free* characterization of the Enneper surface given by the patch (1.1). To achieve this goal, we discover a geometric identity on Enneper's surface. Let's recall

**Definition 1.1 (Angle function on the oriented surface in  $\mathbb{R}^3$ ).** Given a surface  $\Sigma$  in  $\mathbb{R}^3$  oriented by the unit normal vector field  $\mathbf{N}$  and a constant unit vector field  $\mathbf{V}(p) = \mathbf{V}$  in  $\mathbb{R}^3$ , we introduce the angle function  $\mathbf{N}_{\mathbf{V}} : \Sigma \rightarrow [-1, 1]$  by the formula

$$\mathbf{N}_{\mathbf{V}}(p) := \langle \mathbf{N}(p), \mathbf{V} \rangle_{\mathbb{R}^3}, \quad p \in \Sigma,$$

which is the normal component of the vector field  $\mathbf{V}$ .

We observe that the induced unit normal vector field  $\mathbf{N}_{\text{Enn}}$  on the Enneper surface given by

$$\mathbf{X} = \mathbf{X}_{\text{Enn}}(u, v) = \left( \frac{1}{2} \left( u - \frac{u^3}{3} + uv^2 \right), \frac{1}{2} \left( -v + \frac{v^3}{3} - u^2v \right), \frac{1}{2} (u^2 - v^2) \right)$$

reads

$$\mathbf{N}_{\text{Enn}}(u, v) = \frac{1}{\|\mathbf{X}_u \times \mathbf{X}_v\|} \mathbf{X}_u \times \mathbf{X}_v = \frac{1}{1+u^2+v^2} (2u, 2v, -1+u^2+v^2).$$

Taking the downward vector  $-\mathbf{e}_3 = (0, 0, -1)$ , we see that the following quantity is constant:

$$(-\mathcal{K})^{-\frac{1}{4}} (1 + \mathbf{N}_{(-\mathbf{e}_3)}) = \left( \frac{1+u^2+v^2}{2} \right) \cdot \left( 1 + \frac{1-u^2-v^2}{1+u^2+v^2} \right) = 1.$$

It turns out that the constancy of this quantity captures the geometric uniqueness of Enneper's surfaces among minimal surfaces in  $\mathbb{R}^3$ , and that the geometric quantity  $(-\mathcal{K})^{-\frac{1}{4}} (1 + \mathbf{N}_{(-\mathbf{e}_3)})$  motivates the first Chern-Ricci harmonic function on minimal surfaces.

**Theorem 1.2 (The first Chern-Ricci harmonic function and uniqueness of Enneper's surfaces).** Let  $\Sigma$  denote a minimal surface immersed in  $\mathbb{R}^3$  with the Gauss curvature  $\mathcal{K} = \mathcal{K}_{\mathbf{g}_{\Sigma}} < 0$ , unit normal vector field  $\mathbf{N}$ , and angle function  $\mathbf{N}_{\mathbf{V}} \in (-1, 1]$  for a constant unit vector field  $\mathbf{V}$  in  $\mathbb{R}^3$ .

(1) The first Chern-Ricci function  $CR_{\mathbf{V}}^1 := \ln \left( (-\mathcal{K})^{-\frac{1}{4}} (1 + \mathbf{N}_{\mathbf{V}}) \right)$  is harmonic on  $\Sigma$  :

$$\Delta_{\mathbf{g}_{\Sigma}} \ln \left( \frac{1 + \mathbf{N}_{\mathbf{V}}}{(-\mathcal{K})^{\frac{1}{4}}} \right) = 0.$$

(2) If the first Chern-Ricci function is constant, then it should be a part of the Enneper surface, up to isometries and homotheties in  $\mathbb{R}^3$ .

**Theorem 1.3 (Harmonicity of the second Chern-Ricci function and classification of minimal surfaces with constant second Chern-Ricci function).** Let  $\Sigma$  denote a minimal surface in  $\mathbb{R}^3$  with the Gauss curvature  $\mathcal{K} = \mathcal{K}_{\mathbf{g}_{\Sigma}} < 0$ , unit normal vector  $\mathbf{N}$ , and angle function  $\mathbf{N}_{\mathbf{V}} \in (-1, 1)$  for a constant unit vector field  $\mathbf{V}$  in  $\mathbb{R}^3$ .

(1) The second Chern-Ricci function  $CR_V^2 := \ln \left( (-\mathcal{K})^{-\frac{1}{2}} (1 - \mathbf{N}_V^2) \right)$  is harmonic on  $\Sigma$  :

$$\Delta_{\mathbf{g}_\Sigma} \ln \left( \frac{1 - \mathbf{N}_V^2}{(-\mathcal{K})^{\frac{1}{2}}} \right) = 0.$$

(2) Any member of the moduli space of minimal surfaces of constant second Chern-Ricci function is a part of the minimal surface given by Weierstrass data  $\left( \mathbf{g}(\zeta), \frac{1}{\mathbf{g}'(\zeta)} d\zeta \right) = (e^{\alpha(\zeta - \zeta_0)}, e^{-\alpha\zeta} d\zeta)$  for some constants  $\alpha \in \mathbb{C} - \{0\}$  and  $\zeta_0 \in \mathbb{C}$ . The moduli space contains catenoids, helicoids, and their associate families.

**Example 1.4 (The second Chern-Ricci function on helicoids).** Taking the Weierstrass data  $(G(\zeta), \Psi(\zeta)d\zeta) = (e^\zeta, -ie^{-\zeta}d\zeta)$ ,  $\zeta = u + iv \in \mathbb{C}$  gives  $\mathbf{X}(u, v) = (-\sinh u \sin v, \sinh u \cos v, v)$ . The second Chern-Ricci function with respect to the vector field  $\mathbf{V} = \mathbf{e}_3 = (0, 0, 1)$  is constant:

$$CR_{\mathbf{e}_3}^2 = \ln \left( (-\mathcal{K})^{-\frac{1}{2}} (1 - \mathbf{N}_{\mathbf{e}_3}^2) \right) = \ln \left( \left( \frac{1 + |e^\zeta|^2}{2|e^\zeta|} \right)^2 \cdot \left( \frac{2|e^\zeta|}{1 + |e^\zeta|^2} \right)^2 \right) = \ln 1 = 0.$$

*Remark 1.5.* It would be interesting to generalize the Chern-Ricci harmonic functions on minimal hypersurfaces in higher dimensional Euclidean space.

## 2. CONSTRUCTION OF THE FIRST AND SECOND CHERN-RICCI HARMONIC FUNCTIONS

It is a classical fact that the flat points of a minimal surface are isolated, unless itself is flat everywhere. We use the geometric identities due to Chern and Ricci to construct harmonic functions on negatively curved minimal surfaces.

**Theorem 2.1 (Harmonicity of Chern-Ricci functions on minimal surfaces in  $\mathbb{R}^3$ ).** Let  $\Sigma$  denote a minimal surface immersed in  $\mathbb{R}^3$  with the Gauss curvature  $\mathcal{K} = \mathcal{K}_{\mathbf{g}_\Sigma} < 0$ , unit normal vector field  $\mathbf{N}$ , and angle function  $\mathbf{N}_V \in [-1, 1]$  with respect to a constant unit vector field  $\mathbf{V}$  in  $\mathbb{R}^3$ .

(1) When  $-1 < \mathbf{N}_V \leq 1$ , the first Chern-Ricci function  $CR_V^1 = \ln \left( \frac{1 + \mathbf{N}_V}{(-\mathcal{K})^{\frac{1}{4}}} \right)$  is harmonic on  $\Sigma$ .

(2) When  $-1 < \mathbf{N}_V < 1$ , the second Chern-Ricci function  $CR_V^2 = \ln \left( \frac{1 - \mathbf{N}_V^2}{(-\mathcal{K})^{\frac{1}{2}}} \right)$  is harmonic on  $\Sigma$ .

*Proof.* The key idea is to combine two intriguing identities for Gauss curvature function on negatively curved minimal surfaces in  $\mathbb{R}^3$ . On the one hand, in his simple proof of Bernstein's Theorem for entire minimal graphs, Chern [4, Section 4] used the geometric identity

$$\mathcal{K} = \Delta_{\mathbf{g}_\Sigma} \ln(1 + \mathbf{N}_V).$$

On the other hand, Ricci [1, 3, 6, 7, 8] obtained the geometric identity

$$4\mathcal{K} = \Delta_{\mathbf{g}_\Sigma} \ln(-\mathcal{K}), \quad \text{or equivalently,} \quad -\mathcal{K} = \Delta_{\mathbf{g}_\Sigma} \ln \left( \frac{1}{(-\mathcal{K})^{\frac{1}{4}}} \right).$$

Chern's identity and Ricci's identity imply the harmonicity of the first Chern-Ricci function:

$$\Delta_{\mathbf{g}_\Sigma} CR_V^1 = \Delta_{\mathbf{g}_\Sigma} \ln \left( \frac{1 + \mathbf{N}_V}{(-\mathcal{K})^{\frac{1}{4}}} \right) = \Delta_{\mathbf{g}_\Sigma} \ln(1 + \mathbf{N}_V) + \Delta_{\mathbf{g}_\Sigma} \ln \left( \frac{1}{(-\mathcal{K})^{\frac{1}{4}}} \right) = 0.$$

The identity  $-\mathbf{N}_V = \mathbf{N}_{(-V)}$  and the linear combination of two first Chern-Ricci functions imply

$$\Delta_{\mathbf{g}_\Sigma} CR_V^2 = \Delta_{\mathbf{g}_\Sigma} \ln \left( \frac{1 - \mathbf{N}_V^2}{(-\mathcal{K})^{\frac{1}{2}}} \right) = \Delta_{\mathbf{g}_\Sigma} \ln \left( \frac{1 + \mathbf{N}_V}{(-\mathcal{K})^{\frac{1}{4}}} \right) + \Delta_{\mathbf{g}_\Sigma} \ln \left( \frac{1 + \mathbf{N}_{(-V)}}{(-\mathcal{K})^{\frac{1}{4}}} \right) = 0.$$

□

### 3. CLASSIFICATIONS OF MINIMAL SURFACES WITH CONSTANT CHERN-RICCI FUNCTIONS

**Theorem 3.1 (Uniqueness of Enneper's minimal surfaces).** *Let  $\Sigma$  be a minimal surface in  $\mathbb{R}^3$  with the Gauss curvature  $\mathcal{K} = \mathcal{K}_{\mathbf{g}_\Sigma} < 0$  and unit normal vector  $\mathbf{N}$ . Suppose that there exists a constant unit vector field  $\mathbf{V}$  in  $\mathbb{R}^3$  such that  $-1 < \mathbf{N}_{\mathbf{V}} \leq 1$  and that the first Chern-Ricci function  $CR_{\mathbf{V}}^1 = \ln \left( \frac{1 + \mathbf{N}_{\mathbf{V}}}{(-\mathcal{K})^{\frac{1}{4}}} \right)$  is constant. Then, the minimal surface  $\Sigma$  should be a part of Enneper's surface.*

*Proof.* For the simplicity, rotating the coordinate system in  $\mathbb{R}^3$ , we can take the normalization  $\mathbf{V} = -\mathbf{e}_3 = (0, 0, -1)$ . We assume that the first Chern-Ricci function

$$(3.1) \quad \ln \left[ (-\mathcal{K})^{-\frac{1}{4}} (1 + \mathbf{N}_{(-\mathbf{e}_3)}) \right] = C$$

is constant. The key idea is to take the orthogonal lines of curvature on our minimal surface  $\Sigma$  in order to read the information (3.1) in terms of the corresponding Gauss map. We first begin with an arbitrary local conformal coordinate  $w$  on  $\Sigma$  to find the conformal harmonic map :

$$\mathbf{X} = \mathbf{X}(w) = \mathbf{X}(w_0) + \left( \operatorname{Re} \int_{w_0}^w \omega_1, \operatorname{Re} \int_{w_0}^w \omega_2, \operatorname{Re} \int_{w_0}^w \omega_3 \right),$$

where the holomorphic 1-forms  $(\omega_1, \omega_2, \omega_3)$  are given by the Weierstrass data  $(G(w), \Psi(w)dw)$  :

$$(\omega_1, \omega_2, \omega_3) = \left( \frac{1}{2} (1 - G(w)^2) \Psi(w) dw, \frac{i}{2} (1 + G(w)^2) \Psi(w) dw, G(w) \Psi(w) dw \right).$$

We introduce the *isothermic* coordinate  $\zeta$  from the initial conformal coordinate  $w$  by the rule

$$w \mapsto \zeta = \zeta_0 + \int_{w_0}^w \sqrt{G'(w) \Psi(w)} dw,$$

and the mapping  $\mathbf{g}(\zeta) := G(w)$ . Then, the Enneper-Weierstrass representation becomes

$$(3.2) \quad \mathbf{X} = \mathbf{X}(\zeta) = \mathbf{X}(\zeta_0) + \left( \operatorname{Re} \int_{\zeta_0}^{\zeta} \omega_1, \operatorname{Re} \int_{\zeta_0}^{\zeta} \omega_2, \operatorname{Re} \int_{\zeta_0}^{\zeta} \omega_3 \right),$$

where the holomorphic 1-forms  $(\omega_1, \omega_2, \omega_3)$  are given by the Weierstrass data  $(\mathbf{g}(\zeta), \frac{1}{\mathbf{g}'(\zeta)} d\zeta)$  :

$$(3.3) \quad (\omega_1, \omega_2, \omega_3) = \left( \frac{1}{2} \cdot \frac{1 - (\mathbf{g}(\zeta))^2}{\mathbf{g}'(\zeta)} d\zeta, \frac{i}{2} \cdot \frac{1 + (\mathbf{g}(\zeta))^2}{\mathbf{g}'(\zeta)} d\zeta, \frac{\mathbf{g}(\zeta)}{\mathbf{g}'(\zeta)} d\zeta \right).$$

The conformal coordinates  $(u, v) = (\operatorname{Re} \zeta, \operatorname{Im} \zeta)$  give us the orthogonal lines of curvature on  $\Sigma$  :

$$0 = \operatorname{Im} \left( -\mathbf{g}'(\zeta) \cdot \frac{1}{\mathbf{g}'(\zeta)} d\zeta^2 \right) = \operatorname{Im} (-d\zeta^2) = -2 du dv.$$

Using classical formulas (cf. [9, Chapter 9]), we compute Gauss curvature and angle function:

$$(3.4) \quad \mathcal{K} = - \left( \frac{2 |\mathbf{g}'(\zeta)|}{1 + |\mathbf{g}(\zeta)|^2} \right)^4 \quad \text{and} \quad \mathbf{N}_{(-\mathbf{e}_3)} = \frac{1 - |\mathbf{g}(\zeta)|^2}{1 + |\mathbf{g}(\zeta)|^2}.$$

Combining (3.1) and (3.4) gives

$$(3.5) \quad C = \ln \left[ (-\mathcal{K})^{-\frac{1}{4}} (1 + \mathbf{N}_{(-\mathbf{e}_3)}) \right] = \ln \left[ \frac{1 + |\mathbf{g}(\zeta)|^2}{2 |\mathbf{g}'(\zeta)|} \cdot \frac{2}{1 + |\mathbf{g}(\zeta)|^2} \right] = -\operatorname{Re} [\log \mathbf{g}'(\zeta)].$$

By the holomorphicity of the Gauss map  $\mathbf{g}(\zeta)$ , we have  $\mathbf{g}(\zeta) = \alpha(\zeta - \zeta_0)$ . Plugging this into (3.3), the Enneper-Weierstrass representation (3.2) shows that  $\Sigma$  is Enneper's surface.  $\square$

*Remark 3.2.* The associate family deformation of Enneper's surface induces rotations in  $\mathbb{R}^3$ .

**Theorem 3.3 (Classification of minimal surfaces with constant second Chern-Ricci function).** *Let  $\Sigma$  be a minimal surface in  $\mathbb{R}^3$  with the Gauss curvature  $\mathcal{K} = \mathcal{K}_{\mathbf{g}_\Sigma} < 0$  and unit normal vector  $\mathbf{N}$ . Suppose that there exists a constant unit vector field  $\mathbf{V}$  in  $\mathbb{R}^3$  such that  $-1 < \mathbf{N}_\mathbf{V} < 1$  and that the second Chern-Ricci function  $\text{CR}_\mathbf{V}^2 = \ln \left( (-\mathcal{K})^{-\frac{1}{2}} (1 - \mathbf{N}_\mathbf{V}^2) \right)$  is constant. Then,  $\Sigma$  is a part of the minimal surface given by the Enneper-Weierstrass representation*

$$(3.6) \quad \mathbf{X}(\zeta) = \mathbf{X}(\zeta_0) + \left( \text{Re} \int_{\zeta_0}^{\zeta} \omega_1, \text{Re} \int_{\zeta_0}^{\zeta} \omega_2, \text{Re} \int_{\zeta_0}^{\zeta} \omega_3 \right)$$

where the holomorphic 1-forms

$$(3.7) \quad (\omega_1, \omega_2, \omega_3) = \left( \frac{1}{2} \cdot \frac{1 - (\mathbf{g}(\zeta))^2}{\mathbf{g}'(\zeta)} d\zeta, \frac{i}{2} \cdot \frac{1 + (\mathbf{g}(\zeta))^2}{\mathbf{g}'(\zeta)} d\zeta, \frac{\mathbf{g}(\zeta)}{\mathbf{g}'(\zeta)} d\zeta \right).$$

are given by the Weierstrass data  $\left( \mathbf{g}(\zeta), \frac{1}{\mathbf{g}'(\zeta)} d\zeta \right) = (e^{\alpha(\zeta - \zeta_0)}, e^{-\alpha\zeta} d\zeta)$  for some constants  $\alpha \in \mathbb{C} - \{0\}$  and  $\zeta_0 \in \mathbb{C}$ .

*Proof.* For the simplicity, rotating the coordinate system in  $\mathbb{R}^3$ , we can take the normalization  $\mathbf{V} = \mathbf{e}_3 = (0, 0, 1)$ . As in the proof of Theorem 3.1, taking the orthogonal lines of curvature coordinates  $\zeta$  on the minimal surface under the normalization of Hopf differential  $-d\zeta^2$ , we can write the second Chern-Ricci function in terms of the corresponding Gauss map  $\mathbf{g}(\zeta)$ :

$$\text{CR}_{\mathbf{e}_3}^2 = \ln \left( (-\mathcal{K})^{-\frac{1}{2}} (1 - \mathbf{N}_{\mathbf{e}_3}^2) \right) = \ln \left[ \left( \frac{1 + |\mathbf{g}(\zeta)|^2}{2|\mathbf{g}'(\zeta)|} \right)^2 \left( \frac{2|\mathbf{g}(\zeta)|}{1 + |\mathbf{g}(\zeta)|^2} \right)^2 \right] = 2 \ln \left( \frac{|\mathbf{g}(\zeta)|}{|\mathbf{g}'(\zeta)|} \right),$$

or equivalently,

$$(3.8) \quad \text{CR}_{\mathbf{e}_3}^2 = -2 \text{Re} \left[ \log \frac{\mathbf{g}'(\zeta)}{\mathbf{g}(\zeta)} \right] = -2 \text{Re} \left[ \log (\log \mathbf{g}(\zeta))' \right].$$

Since the function  $\text{CR}_{\mathbf{e}_3}^2$  is constant, by the holomorphicity of  $\mathbf{g}(\zeta)$ , we have the Weierstrass data

$$\left( \mathbf{g}(\zeta), \frac{1}{\mathbf{g}'(\zeta)} d\zeta \right) = (e^{\alpha(\zeta - \zeta_0)}, e^{-\alpha\zeta} d\zeta).$$

□

*Remark 3.4 (Examples of minimal surfaces with constant second Chern-Ricci function).* The moduli space in Theorem 3.3 contains catenoids [2, Section 2.1.2] and helicoids [2, Section 2.1.4]. In fact, under the transformation  $\zeta \mapsto z := e^{\alpha(\zeta - \zeta_0)}$ , we obtain the Weierstrass data

$$\left( \mathbf{g}(\zeta), \frac{1}{\mathbf{g}'(\zeta)} d\zeta \right) = (e^{\alpha(\zeta - \zeta_0)}, e^{-\alpha\zeta} d\zeta) = \left( z, \frac{e^{-\alpha\zeta_0}}{\alpha} \cdot \frac{1}{z^2} dz \right),$$

which recovers the associate family of helicoids.

#### APPENDIX. FOUR HOLOMORPHIC QUADRATIC DIFFERENTIALS ON MINIMAL SURFACES IN $\mathbb{R}^3$

We summarize definitions of holomorphic quadratic differentials on minimal surfaces in  $\mathbb{R}^3$ . Throughout this section, as in the proofs of Theorem 3.1 and 3.3, we use the orthogonal lines of curvature coordinates  $\zeta$  on the negatively curved minimal surface  $\Sigma$  with the normalization of Hopf's holomorphic differential  $\mathcal{Q} = -\frac{1}{2}d\zeta^2$ . The minimal surface  $\Sigma$  is parameterized by

$$\mathbf{X}(\zeta) = \mathbf{X}(\zeta_0) + \left( \text{Re} \int_{\zeta_0}^{\zeta} \omega_1, \text{Re} \int_{\zeta_0}^{\zeta} \omega_2, \text{Re} \int_{\zeta_0}^{\zeta} \omega_3 \right),$$

where we have the holomorphic 1-forms  $(\omega_1, \omega_2, \omega_3) = \left( \frac{1}{2} \cdot \frac{1 - (\mathbf{g}(\zeta))^2}{\mathbf{g}'(\zeta)}, \frac{i}{2} \cdot \frac{1 + (\mathbf{g}(\zeta))^2}{\mathbf{g}'(\zeta)}, \frac{\mathbf{g}(\zeta)}{\mathbf{g}'(\zeta)} \right) d\zeta$ .

**I. Bernstein-Mettler's entropy differential** [3]. Applying the variational structure of Ricci's intrinsic condition induced from Hamilton's entropy functional for Ricci flow, Bernstein and Mettler constructed the entropy differential. In [3, Corollary A.2], they observed that, if the minimal surface  $\Sigma$  is an Enneper surface, the conformally changed metric  $(-\mathcal{K}_{\text{Enn}})^{\frac{3}{4}} \mathbf{g}_{\text{Enn}}$  recovers the positively curved cigar soliton, and also proved that this property characterizes Enneper surfaces among minimal surfaces in  $\mathbb{R}^3$ . The formula in [3, Proposition 3.2] implies that the Schwarzian derivative of the Gauss map  $\mathbf{g}(\zeta)$  realizes the holomorphic quadratic differential

$$(3.9) \quad \mathbf{Sg}(\zeta) d\zeta^2 := \left[ \left( \frac{\mathbf{g}''(\zeta)}{\mathbf{g}'(\zeta)} \right)' - \frac{1}{2} \left( \frac{\mathbf{g}''(\zeta)}{\mathbf{g}'(\zeta)} \right)^2 \right] d\zeta^2.$$

We would like to add that the Schwarzian derivative of the gauss map realizes the squared complex curvature of the lifted holomorphic null curve from the minimal surface [5, Section 3].

**II. Induced holomorphic quadratic differential from Chern-Ricci functions.** By the identity (3.5), the holomorphicity of the corresponding Gauss map  $\mathbf{g}(\zeta)$  also implies the harmonicity of the first Chern-Ricci function

$$\text{CR}_{(-\mathbf{e}_3)}^1 = \ln \left[ (-\mathcal{K})^{-\frac{1}{4}} (1 + \mathbf{N}_{(-\mathbf{e}_3)}) \right] = -\text{Re} [\log \mathbf{g}'(\zeta)].$$

The harmonic function  $-\text{CR}_{(-\mathbf{e}_3)}^1$  induces the holomorphic quadratic differential

$$(3.10) \quad \mathcal{Q}_1 = (\log \mathbf{g}'(\zeta))'' d\zeta^2 = \left( \frac{\mathbf{g}''(\zeta)}{\mathbf{g}'(\zeta)} \right)' d\zeta^2.$$

By the identity (3.8), the holomorphicity of the Gauss map  $\mathbf{g}(\zeta)$  also implies the harmonicity of the second Chern-Ricci function

$$\text{CR}_{\mathbf{e}_3}^2 = \ln \left( (-\mathcal{K})^{-\frac{1}{2}} (1 - \mathbf{N}_{\mathbf{e}_3}^2) \right) = -2 \text{Re} [\log (\log \mathbf{g}(\zeta))'].$$

The harmonic function  $-\frac{1}{2}\text{CR}_{\mathbf{e}_3}^2$  induces the holomorphic quadratic differential

$$(3.11) \quad \mathcal{Q}_2 = (\log (\log \mathbf{g}(\zeta)))'' d\zeta^2 = \left[ \left( \frac{\mathbf{g}''(\zeta)}{\mathbf{g}'(\zeta)} \right)' - \left( \frac{\mathbf{g}'(\zeta)}{\mathbf{g}(\zeta)} \right)' \right] d\zeta^2.$$

## REFERENCES

- [1] W. Blaschke, *Einführung in die Differentialgeometrie*, Springer, Berlin, 1950.
- [2] F. Brito, M. L. Leite, V. Neto, *Liouville's formula under the viewpoint of minimal surfaces*, Commun. Pure Appl. Anal. **3** (2004), no. 1, 41–51.
- [3] J. Bernstein, T. Mettler, *Characterizing classical minimal surfaces via a new meromorphic quadratic differential*, arXiv preprint, <https://arxiv.org/abs/1301.1663v3>.
- [4] S. S. Chern, *Simple proofs of two theorems on minimal surfaces*, Enseign. Math. (2) **15** (1969), 53–61.
- [5] H. Gollek, *Deformations of minimal surfaces of  $\mathbb{R}^3$  containing planar geodesics*, The 18th Winter School Geometry and Physics (Srn, 1998), Rend. Circ. Mat. Palermo (2) Suppl. **59** (1999), 143–153.
- [6] H. B. Lawson, *Complete minimal surfaces in  $\mathbb{S}^3$* , Ann. of Math. (2), **92** (1970), 335–374.
- [7] H. B. Lawson, *Some intrinsic characterizations of minimal surfaces*, J. Analyse Math. **24** (1971), 151–161.
- [8] A. Moroianu, S. Moroianu, *Ricci surfaces*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **14** (2015), no. 4, 1093–1118.
- [9] R. Osserman, *A survey of minimal surfaces*. Second edition. Dover Publications, Inc., New York, 1986.
- [10] M. Weber, *The Enneper Surface*, <http://www.indiana.edu/~minimal/archive/Classical/Classical/Enneper/web/index.html>.

CENTER FOR MATHEMATICAL CHALLENGES, KOREA INSTITUTE FOR ADVANCED STUDY, HOEGIRO 85, DONGDAEMUN-GU, SEOUL 02455, KOREA

E-mail address: momentmaplee@gmail.com